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Some Infinite (Singly, Doubly and Triply) Sums

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Abstract

In this article some infinite (singly, doubly and triply) sums are reported. Some of them are shown as follows for example. That is,

$$(i) \quad \sum_{k=1}^{\infty} \frac{1}{k(k+m)(k+m+1) \cdots (k+m+n)} = \frac{(m-1)!}{(n+m)!} \sum_{k=1}^m \frac{1}{(n+k)},$$

(Singly infinite sum)

$$(ii) \quad \sum_{n=0}^{\infty} \sum_{k=1}^{\infty} \frac{\left[(m-1)! \cdot \left(\sum_{k=1}^m (n+k)^{-1} \right) \right]^{-1}}{k(k+m)(k+m+1) \cdots (k+m+n)} = e^{-\sum_{k=0}^{m-1} \frac{1}{k!}}$$

(Doubly infinite sum)

$$(iii) \quad \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \sum_{k=1}^{\infty} \frac{\left[(m-1)! \cdot (n+m)! \left(\sum_{k=1}^m (n+k)^{-1} \right) \right]^{-1}}{k(k+m)(k+m+1) \cdots (k+m+n)} = 1.59063 \cdots,$$

(Triply infinite sum)

where $m \in \mathbb{Z}^+$ and $n \in \mathbb{Z}_0^+$.

§ 0. Introduction (Definition of Fractional Calculus)

(I) Definition. (by K. Nishimoto) ([1] Vol. 1)

Let $D = \{D_-, D_+\}$, $C = \{C_-, C_+\}$,

C_- be a curve along the cut joining two points z and $-\infty + i \operatorname{Im}(z)$,

C_+ be a curve along the cut joining two points z and $\infty + i \operatorname{Im}(z)$,

D_- be a domain surrounded by C_- , D_+ be a domain surrounded by C_+ .

(Here D contains the points over the curve C).

Moreover, let $f = f(z)$ be a regular function in D ($z \in D$),

$$f_v = (f)_v = {}_C(f)_v = \frac{\Gamma(v+1)}{2\pi i} \int_C \frac{f(\xi)}{(\xi-z)^{v+1}} d\xi \quad (v \notin \mathbb{Z}), \quad (1)$$

$$(f)_{-m} = \lim_{v \rightarrow -m} (f)_v \quad (m \in \mathbb{Z}^+), \quad (2)$$

where $-\pi \leq \arg(\zeta - z) \leq \pi$ for C_- , $0 \leq \arg(\zeta - z) \leq 2\pi$ for C_+ ,

$\zeta \neq z$, $z \in C$, $\nu \in \mathbb{R}$, Γ ; Gamma function,

then $(f)_\nu$ is the fractional differentiation of arbitrary order ν (derivatives of order ν for $\nu > 0$, and integrals of order $-\nu$ for $\nu < 0$), with respect to z , of the function f , if $|(f)_\nu| < \infty$.

(II) On the fractional calculus operator N^ν [3]

Theorem A. Let fractional calculus operator (Nishimoto's Operator) N^ν be

$$N^\nu = \left(\frac{\Gamma(\nu+1)}{2\pi i} \int_C \frac{d\zeta}{(\zeta - z)^{\nu+1}} \right) \quad (\nu \notin \mathbb{Z}), \quad [\text{Refer to (1)}] \quad (3)$$

$$\text{with} \quad N^{-m} = \lim_{\nu \rightarrow -m} N^\nu \quad (m \in \mathbb{Z}^+), \quad (4)$$

and define the binary operation \circ as

$$N^\beta \circ N^\alpha f = N^\beta N^\alpha f = N^\beta(N^\alpha f) \quad (\alpha, \beta \in \mathbb{R}), \quad (5)$$

then the set

$$\{N^\nu\} = \{N^\nu | \nu \in \mathbb{R}\} \quad (6)$$

is an Abelian product group (having continuous index ν) which has the inverse transform operator $(N^\nu)^{-1} = N^{-\nu}$ to the fractional calculus operator N^ν , for the function f such that $f \in F = \{f; 0 \neq |f_\nu| < \infty, \nu \in \mathbb{R}\}$, where $f = f(z)$ and $z \in C$. (vis. $-\infty < \nu < \infty$).

(For our convenience, we call $N^\beta \circ N^\alpha$ as product of N^β and N^α .)

Theorem B. "F.O.G. $\{N^\nu\}$ " is an "Action product group which has continuous index ν " for the set of F . (F.O.G.; Fractional calculus operator group)

(III) Lemma [1]

$$(i) \quad ((z-c)^b)_\alpha = e^{-i\pi b} \frac{\Gamma(\alpha-b)}{\Gamma(-b)} (z-c)^{b-\alpha} \quad \left(\left| \frac{\Gamma(\alpha-b)}{\Gamma(-b)} \right| < \infty \right).$$

$$(ii) \quad (\log(z-c))_\alpha = -e^{-i\pi\alpha} \Gamma(\alpha) (z-c)^{-\alpha} \quad (|\Gamma(\alpha)| < \infty),$$

$$(iii) \quad (u \cdot v)_\alpha \equiv \sum_{k=0}^{\infty} \frac{\Gamma(\alpha+1)}{k! \Gamma(\alpha+1-k)} u_{\alpha-k} v_k \quad \begin{pmatrix} u = u(z) \\ v = v(z) \end{pmatrix},$$

where $z \neq c$ in (i) and (ii).

§ 1. Singly Infinite Sums

Theorem 1. We have

$$\sum_{k=1}^{\infty} \frac{1}{k(k+m)(k+m+1) \cdots (k+m+n)} = \frac{(m-1)!}{(n+m)!} \sum_{k=1}^m \frac{1}{n+k} \quad (1)$$

where $m \in \mathbb{Z}^+$, $n \in \mathbb{Z}_0^+ (= \mathbb{Z}^+ \cup \{0\})$.

Proof. We have

$$(z^n \cdot \log az)_{-m} = \sum_{k=0}^{\infty} \frac{\Gamma(-m+1)}{k! \Gamma(-m+1-k)} (z^n)_{-m-k} (\log az)_k \quad (az \neq 0) \quad (2)$$

$$= (z^n)_{-m} \cdot \log az + \sum_{k=1}^{\infty} \frac{(-1)^k \cdot m(m+1) \cdots (m+k-1)}{k!} (z^n)_{-m-k} (\log az)_k \quad (3)$$

$$= e^{i\pi m} \frac{\Gamma(-n-m)}{\Gamma(-n)} z^{n+m} \cdot \log az + \sum_{k=1}^{\infty} \frac{(-1)^k \cdot m(m+1) \cdots (m+k-1)}{k!} \\ \times e^{i\pi(k+m)} \frac{\Gamma(-n-m-k)}{\Gamma(-n)} z^{n+m+k} \cdot (-e^{-i\pi k} \Gamma(k) z^{-k}) \quad (4)$$

$$= \frac{1}{(n+1)(n+2) \cdots (n+m)} z^{n+m} \cdot \log az \\ - z^{n+m} \sum_{k=1}^{\infty} \frac{(-1)^k \cdot m(m+1) \cdots (m+k-1)}{k} \cdot \frac{e^{i\pi m} (-1)^{m+k}}{(n+1)(n+2) \cdots (n+m+k)} \quad (5)$$

$$= \frac{n!}{(n+m)!} z^{n+m} \log az - \frac{n!}{(m-1)!} z^{n+m} Q_{m,n}, \quad (6)$$

that is,

$$(z^n \cdot \log az)_{-m} = \frac{n!}{(n+m)!} z^{n+m} \log az - \frac{n!}{(m-1)!} z^{n+m} Q_{m,n}, \quad (7)$$

where

$$Q_{m,n} = \sum_{k=1}^{\infty} \frac{1}{k(k+m)(k+m+1) \cdots (k+m+n)} = \sum_{k=1}^{\infty} \frac{(m+k-1)!}{(k+m+n)! k}. \quad (8)$$

Therefore, operating N-fractional calculus operator N^m to the both sides of (7) (making m th order derivative of both sides) we obtain

$$((z^n \cdot \log az)_{-m})_m = \frac{n!}{(n+m)!} (z^{n+m} \log az)_m - \frac{n!}{(m-1)!} (z^{n+m})_m Q_{m,n} \quad (9)$$

hence

$$z^n \cdot \log az \\ = z^n \log az + \frac{n!}{(n+m)!} \left\{ (n+1)(n+2) \cdots (n+m) \left(\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+m} \right) \right\} z^n \\ - \frac{(n+m)!}{(m-1)!} z^n Q_{m,n} \quad (10)$$

under the conditions.

From (10) we have

$$Q_{m,n} = \frac{(m-1)! \cdot n!}{\{(n+m)!\}^2} \left\{ \prod_{k=1}^m (n+k) \right\} \left(\sum_{k=1}^m \frac{1}{n+k} \right) = \frac{(m-1)!}{(n+m)!} \sum_{k=1}^m \frac{1}{n+k} . \quad (11)$$

Therefore, we have (1) from (11) clearly, under the conditions.

Note 1. We calculate as, for example,

$$\frac{\Gamma(-1)}{\Gamma(-2)} = \lim_{\alpha \rightarrow 0} \frac{\Gamma(\alpha-1)}{\Gamma(\alpha-2)} = \lim_{\alpha \rightarrow 0} \frac{\Gamma(\alpha-2+1)}{\Gamma(\alpha-2)} = \lim_{\alpha \rightarrow 0} \frac{(\alpha-2)\Gamma(\alpha-2)}{\Gamma(\alpha-2)} = -2. \quad (12)$$

However we calculate this as

$$\Gamma(-1)/\Gamma(-2) = \Gamma(1-2)/\Gamma(-2) = (-2)\Gamma(-2)/\Gamma(-2) = -2, \quad (13)$$

for our convenience, using the relationship $\Gamma(z+1) = z\Gamma(z)$ for Gamma function.

§ 2. Doubly Infinite Sums

Theorem 2. Let

$$A(k, m, n) = \frac{\left[(m-1)! \left(\sum_{k=1}^m (n+k)^{-1} \right) \right]^{-1}}{k(k+m)(k+m+1) \cdots (k+m+n)} . \quad (1)$$

We have then

$$(i) \quad \sum_{n=0}^{\infty} \sum_{k=1}^{\infty} A(k, m, n) = e - \sum_{k=0}^{m-1} \frac{1}{k!} \quad (2)$$

$$(ii) \quad \sum_{m=1}^{\infty} \sum_{k=1}^{\infty} A(k, m, n) = e - \sum_{k=0}^n \frac{1}{k!} \quad (3)$$

and

$$(iii) \quad \sum_{n=0}^{\infty} \sum_{k=1}^{\infty} A(k, p+1, n) = \sum_{m=1}^{\infty} \sum_{k=1}^{\infty} A(k, m, p) \quad (4)$$

where $m \in \mathbb{Z}^+$, $n \in \mathbb{Z}_0^+ (= \mathbb{Z}^+ \cup \{0\})$, $p \in \mathbb{Z}_0^+$ and $e = 2.71828 \dots$.

Proof of (i). Now we have

$$\sum_{k=1}^{\infty} \frac{\left[(m-1)! \left(\sum_{k=1}^m (n+k)^{-1} \right) \right]^{-1}}{k(k+m)(k+m+1) \cdots (k+m+n)} = \frac{1}{(m+n)!} \quad (5)$$

from Theorem 1.

We have then

$$\sum_{k=1}^{\infty} A(k, m, n) = \frac{1}{(m+n)!} \quad (6)$$

from (5) and (1).

Therefore, we have

$$\sum_{n=0}^{\infty} \sum_{k=1}^{\infty} A(k, m, n) = \sum_{n=0}^{\infty} \frac{1}{(m+n)!} . \quad (7)$$

We obtain (2) from (7) under the conditions, since

$$e = \sum_{k=0}^{m-1} \frac{1}{k!} + \sum_{k=m}^{\infty} \frac{1}{k!} . \quad (8)$$

Proof of (ii). We have

$$\sum_{m=1}^{\infty} \sum_{k=1}^{\infty} A(k, m, n) = \sum_{m=1}^{\infty} \frac{1}{(m+n)!} . \quad (9)$$

Therefore, we obtain (3) under the conditions from (9), since

$$e = \sum_{k=0}^n \frac{1}{k!} + \sum_{k=n+1}^{\infty} \frac{1}{k!} . \quad (10)$$

Proof of (iii). We have

$$\sum_{n=0}^{\infty} \sum_{k=1}^{\infty} A(k, p+1, n) = e - \sum_{k=0}^p \frac{1}{k!} \quad (11)$$

and

$$\sum_{m=1}^{\infty} \sum_{k=1}^{\infty} A(k, m, p) = e - \sum_{k=0}^p \frac{1}{k!} \quad (12)$$

from (2) and (3) respectively.

Therefore, we obtain (4) from (11) and (12), under the conditions.

Theorem 3. Let

$$B(k, m, n) = \frac{\left[(m-1)! \cdot (m+n)! \left(\sum_{k=1}^m (n+k)^{-1} \right) \right]^{-1}}{k(k+m)(k+m+1) \cdots (k+m+n)} . \quad (13)$$

We have then

$$(i) \quad \sum_{n=0}^{\infty} \sum_{k=1}^{\infty} B(k, m, n) = \sum_{k=m}^{\infty} \frac{1}{(k!)^2} \quad (14)$$

$$(ii) \quad \sum_{m=1}^{\infty} \sum_{k=1}^{\infty} B(k, m, n) = \sum_{k=n}^{\infty} \frac{1}{\{(k+1)!\}^2} \quad (15)$$

and

$$(iii) \quad \sum_{n=0}^{\infty} \sum_{k=1}^{\infty} B(k, p+1, n) = \sum_{m=1}^{\infty} \sum_{k=1}^{\infty} B(k, m, p) \quad (16)$$

where $m \in \mathbb{Z}^+$, $n \in \mathbb{Z}_0^+ (= \mathbb{Z}^+ \cup \{0\})$ and $p \in \mathbb{Z}_0^+$.

Proof of (i) and (ii). Now we have

$$\sum_{k=1}^{\infty} \frac{\left[(m-1)! \cdot (m+n)! \left(\sum_{k=1}^m (n+k)^{-1} \right) \right]^{-1}}{k(k+m)(k+m+1) \cdots (k+m+n)} = \frac{1}{\{(m+n)!\}^2} \quad (17)$$

from Theorem 1.

We have then

$$\sum_{k=1}^{\infty} B(k, m, n) = \frac{1}{\{(m+n)!\}^2} \quad (18)$$

from (13) and (17).

Therefore, we have

$$\sum_{n=0}^{\infty} \sum_{k=1}^{\infty} B(k, m, n) = \sum_{n=0}^{\infty} \frac{1}{\{(m+n)!\}^2} \quad (19)$$

$$= \sum_{k=m}^{\infty} \frac{1}{(k!)^2} \quad (14)$$

and

$$\sum_{m=1}^{\infty} \sum_{k=1}^{\infty} B(k, m, n) = \sum_{m=1}^{\infty} \frac{1}{\{(m+n)!\}^2} \quad (20)$$

$$= \sum_{k=n}^{\infty} \frac{1}{\{(k+1)!\}^2} \quad (15)$$

from (18) respectively.

Proof of (iii). We have

$$\sum_{n=0}^{\infty} \sum_{k=1}^{\infty} B(k, p+1, n) = \sum_{k=p+1}^{\infty} \frac{1}{(k!)^2} \quad (21)$$

and

$$\sum_{m=1}^{\infty} \sum_{k=1}^{\infty} B(k, m, p) = \sum_{k=p}^{\infty} \frac{1}{\{(k+1)!\}^2} \quad (22)$$

from (14) and (15) respectively.

Therefore, we obtain (16) from (21) and (22), under the conditions.

§ 3. A Triply Infinite Sums

Theorem 4. We have the triple infinite sum

$$\sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \sum_{k=1}^{\infty} \frac{\left[(m-1)! \cdot (n+m)! \left(\sum_{k=1}^m (n+k)^{-1} \right) \right]^{-1}}{k(k+m)(k+m+1) \cdots (k+m+n)} = 1.59063 \dots, \quad (1)$$

where $m \in \mathbb{Z}^+$ and $n \in \mathbb{Z}_0^+$.

Proof. We have

$$\sum_{k=1}^{\infty} B(k, m, n) = \frac{1}{\{(m+n)!\}^2} \quad (2)$$

then

$$\sum_{m=1}^{\infty} \sum_{k=1}^{\infty} B(k, m, n) = \sum_{m=1}^{\infty} \frac{1}{\{(m+n)!\}^2} \quad (3)$$

$$= \sum_{k=n}^{\infty} \frac{1}{\{(k+1)!\}^2}, \quad (4)$$

where $B(k, m, n)$ is the one shown by § 1. (13).

Therefore, we have

$$\sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \sum_{k=1}^{\infty} B(k, m, n) = \sum_{n=0}^{\infty} \sum_{k=n}^{\infty} \frac{1}{\{(k+1)!\}^2} \quad (5)$$

$$= \sum_{n=0}^{\infty} \left\{ \frac{1}{\{(n+1)!\}^2} + \frac{1}{\{(n+2)!\}^2} + \frac{1}{\{(n+3)!\}^2} + \dots \right\} \quad (6)$$

$$= \sum_{n=0}^{\infty} \frac{1}{\{(n+1)!\}^2} + \sum_{n=0}^{\infty} \frac{1}{\{(n+2)!\}^2} + \sum_{n=0}^{\infty} \frac{1}{\{(n+3)!\}^2} + \dots \quad (7)$$

$$= 1.279584\dots + 0.279584\dots + 0.029584\dots$$

$$+ 0.001806\dots + 0.000070\dots + 0.0000011\dots$$

$$+ 0.0000000\dots + \dots \quad (8)$$

$$= 1.59063\dots \quad (1)$$

from (4).

§ 4. Illustrative Examples of Theorems 1, 2 and 3

(I) Examples of Theorem 1.

We obtain

$$Q_{1,0} = \sum_{k=1}^{\infty} \frac{1}{k(k+1)} = 1 \quad ([17] \text{ p. 656, } [18] \text{ p. 42.}) \quad (1)$$

$$Q_{1,1} = \sum_{k=1}^{\infty} \frac{1}{k(k+1)(k+2)} = \frac{1}{2!2} \quad ([17] \text{ p. 666, } [18] \text{ p. 43.}) \quad (2)$$

$$Q_{1,2} = \sum_{k=1}^{\infty} \frac{1}{k(k+1)(k+2)(k+3)} = \frac{1}{3!3} \quad ([17] \text{ p. 675, } [18] \text{ p. 43.}) \quad (3)$$

$$Q_{1,3} = \sum_{k=1}^{\infty} \frac{1}{k(k+1)(k+2)(k+3)(k+4)} = \frac{1}{4!4} \quad (4)$$

$$Q_{1,4} = \sum_{k=1}^{\infty} \frac{1}{k(k+1)(k+2)(k+3)(k+4)(k+5)} = \frac{1}{5!5} = \frac{1}{600} \quad (5)$$

for $m=1$,

$$Q_{2,0} = \sum_{k=1}^{\infty} \frac{1}{k(k+2)} = \frac{3}{4} \quad ([17] \text{ p. 656, } [18] \text{ p. 45.}) \quad (6)$$

$$Q_{2,1} = \sum_{k=1}^{\infty} \frac{1}{k(k+2)(k+3)} = \frac{5}{(3!)^2} = \frac{5}{36} \quad ([17] \text{ p. 666.}) \quad (7)$$

$$Q_{2,2} = \sum_{k=1}^{\infty} \frac{1}{k(k+2)(k+3)(k+4)} = \frac{2!7}{(4!)^2} = \frac{7}{288} \quad (8)$$

$$Q_{2,3} = \sum_{k=1}^{\infty} \frac{1}{k(k+2)(k+3)(k+4)(k+5)} = \frac{3!9}{(5!)^2} = \frac{3}{800} \quad (9)$$

for $m = 2$,

$$Q_{3,0} = \sum_{k=1}^{\infty} \frac{1}{k(k+3)} = \frac{11}{18} \quad ([17] \text{ p. 656.}) \quad (10)$$

$$Q_{3,1} = \sum_{k=1}^{\infty} \frac{1}{k(k+3)(k+4)} = \frac{2 \cdot 26}{(4!)^2} = \frac{13}{144} \quad (11)$$

$$Q_{3,2} = \sum_{k=1}^{\infty} \frac{1}{k(k+3)(k+4)(k+5)} = \frac{2!2 \cdot 47}{(5!)^2} = \frac{47}{3600} \quad (12)$$

$$Q_{3,3} = \sum_{k=1}^{\infty} \frac{1}{k(k+3)(k+4)(k+5)(k+6)} = \frac{3!2 \cdot 74}{(6!)^2} = \frac{37}{21600} \quad (13)$$

for $m = 3$,

$$Q_{4,0} = \sum_{k=1}^{\infty} \frac{1}{k(k+4)} = \frac{25}{48} \quad ([17] \text{ p. 656.}) \quad (14)$$

$$Q_{4,1} = \sum_{k=1}^{\infty} \frac{1}{k(k+4)(k+5)} = \frac{3! \cdot 154}{(5!)^2} = \frac{77}{1200} \quad (15)$$

$$Q_{4,2} = \sum_{k=1}^{\infty} \frac{1}{k(k+4)(k+5)(k+6)} = \frac{2!3! \cdot 342}{(6!)^2} = \frac{19}{2400} \quad (16)$$

$$Q_{4,3} = \sum_{k=1}^{\infty} \frac{1}{k(k+4)(k+5)(k+6)(k+7)} = \frac{(3!)^2 638}{(7!)^2} = \frac{319}{352800} \quad (17)$$

for $m = 4$,

$$Q_{5,0} = \sum_{k=1}^{\infty} \frac{1}{k(k+5)} = \frac{137}{300} \quad (18)$$

$$Q_{5,1} = \sum_{k=1}^{\infty} \frac{1}{k(k+5)(k+6)} = \frac{4! \cdot 1044}{(6!)^2} = \frac{29}{600} \quad (19)$$

for $m = 5$, and

$$Q_{m,0} = \sum_{k=1}^{\infty} \frac{1}{k(k+m)} = \frac{1}{m} \sum_{k=1}^m \frac{1}{k} \quad ([17] \text{ p. 656 }). \quad (20)$$

(II) Examples of Theorem 2. (i);

$$1) \quad \sum_{n=0}^{\infty} \sum_{k=1}^{\infty} \frac{n+1}{k(k+1)(k+2)\cdots(k+1+n)} = 1.71828 \dots \quad (21)$$

(Set $m = 1$ in § 2. (2).)

$$2) \quad \sum_{n=0}^{\infty} \sum_{k=1}^{\infty} \frac{(n+1)(n+2) \cdot (2n+3)^{-1}}{k(k+2)(k+3)\cdots(k+2+n)} = 0.71828 \dots \quad (22)$$

(Set $m = 2$ in § 2. (2).)

$$3) \quad \sum_{n=0}^{\infty} \sum_{k=1}^{\infty} \frac{[2 \sum_{k=1}^3 (n+k)^{-1}]^{-1}}{k(k+3)(k+4)\cdots(k+3+n)} = 0.21828 \dots \quad (23)$$

(Set $m = 3$ in § 2. (2).)

Note. Indeed, having $m = 1$ we obtain

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)} = 1 \quad (\text{from } Q_{1,0}) \quad ([17] \text{ p. 656, } [18] \text{ p. 42.}) \quad (24)$$

$$\sum_{k=1}^{\infty} \frac{2}{k(k+1)(k+2)} = \frac{1}{2!} \quad (\text{from } Q_{1,1}) \quad ([17] \text{ p. 666, } [18] \text{ p. 43.}) \quad (25)$$

$$\sum_{k=1}^{\infty} \frac{3}{k(k+1)(k+2)(k+3)} = \frac{1}{3!} \quad (\text{from } Q_{1,2}) \quad ([17] \text{ p. 675, } [18] \text{ p. 43.}) \quad (26)$$

$$\sum_{k=1}^{\infty} \frac{4}{k(k+1)(k+2)(k+3)(k+4)} = \frac{1}{4!} \quad (\text{from } Q_{1,3}) \quad (27)$$

$$\dots\dots\dots (28)$$

$$\sum_{k=1}^{\infty} \frac{n+1}{k(k+1)(k+2)\cdots(k+1+n)} = \frac{1}{(n+1)!} \quad (\text{from } Q_{1,n}) \quad (29)$$

$$\dots\dots\dots (30)$$

from Theorem 1 in § 1.

Therefore, by the summing up from (24) to (30) we obtain

$$\sum_{n=0}^{\infty} \sum_{k=1}^{\infty} \frac{n+1}{k(k+1)(k+2)\cdots(k+1+n)} = \sum_{n=0}^{\infty} \frac{1}{(n+1)!} \quad (31)$$

$$= e - 1 = 1.71828 \dots, \quad (32)$$

clearly, for example.

(III) Examples of Theorem 2. (ii);

$$1) \quad \sum_{m=1}^{\infty} \sum_{k=1}^{\infty} \frac{[(m-1)! \cdot (\sum_{k=1}^m k^{-1})]^{-1}}{k(k+m)} = 1.71828 \dots \quad (33)$$

(Set $n = 0$ in § 2. (3).)

$$2) \quad \sum_{m=1}^{\infty} \sum_{k=1}^{\infty} \frac{[(m-1)! \cdot (\sum_{k=1}^m (1+k)^{-1})]^{-1}}{k(k+m)(k+m+1)} = 0.71828 \dots \quad (34)$$

(Set $n = 1$ in § 2. (3).)

$$3) \quad \sum_{m=1}^{\infty} \sum_{k=1}^{\infty} \frac{[(m-1)! \cdot (\sum_{k=1}^m k^{-1})]^{-1}}{k(k+m)} = 0.21828 \dots \quad (35)$$

(Set $n = 2$ in § 2. (3).)

(IV) Examples of Theorem 3 (i);

$$1) \quad \sum_{n=0}^{\infty} \sum_{k=1}^{\infty} \frac{[n!]^{-1}}{k(k+1)(k+2)\cdots(k+1+n)} = 1.279584 \dots \quad (36)$$

(Set $m = 1$ in § 2. (14).)

$$2) \quad \sum_{n=0}^{\infty} \sum_{k=1}^{\infty} \frac{[n! \cdot (2n+3)]^{-1}}{k(k+2)(k+3)\cdots(k+2+n)} = 0.279584 \dots \quad (37)$$

(Set $m = 2$ in § 2. (14).)

$$3) \quad \sum_{n=0}^{\infty} \sum_{k=1}^{\infty} \frac{[n! \cdot 2(3n^2 + 12n + 11)]^{-1}}{k(k+3)(k+4)\cdots(k+3+n)} = 0.029584 \dots \quad (38)$$

(Set $m = 3$ in § 2. (14).)

(V) Examples of Theorem 3. (ii);

$$1) \quad \sum_{m=1}^{\infty} \sum_{k=1}^{\infty} \frac{[(m-1)! \cdot m! \cdot (\sum_{k=1}^m k^{-1})]^{-1}}{k(k+m)} = 1.279584 \dots \quad (39)$$

(Set $n = 0$ in § 2. (15).)

$$2) \quad \sum_{m=1}^{\infty} \sum_{k=1}^{\infty} \frac{[(m-1)! \cdot (1+m)! \cdot (\sum_{k=1}^m (1+k)^{-1})]^{-1}}{k(k+m)(k+m+1)} = 0.279584 \dots \quad (40)$$

(Set $n=1$ in § 2. (15).)

$$3) \quad \sum_{m=1}^{\infty} \sum_{k=1}^{\infty} \frac{[(m-1)! \cdot (2+m)! \cdot (\sum_{k=1}^m (2+k)^{-1})]^{-1}}{k(k+m)(k+m+1)(k+m+2)} = 0.21828 \dots \quad (41)$$

(Set $n=2$ in § 2. (15).)

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$$Q_{m,n} = \sum_{k=1}^{\infty} \frac{1}{k(k+m)(k+m+1) \cdots (k+m+n)} \quad (n \in \mathbb{Z}^+ \cup \{0\}, m \in \mathbb{Z}^+)$$
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$$\sum_{k=1}^{\infty} \frac{(n-1)! \cdot 2^{n-1}}{\prod_{k=0}^n (2k+3)} \quad \text{and} \quad \sum_{k=1}^{\infty} \frac{(n-1)! \cdot (n+1) \cdot 2^{n-1}}{\prod_{k=0}^{n+1} (2k+3)}$$
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$$R_{m,\beta} = (-1)^n \sum_{k=1}^{\infty} \frac{(m+k-1)! \cdot (-1)^k}{(m-1)! k} \cdot \frac{\Gamma(-m-k-\beta)}{\Gamma(-\beta)}$$
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